Part of
Ertel, P. K., & Ridgley, R. (Eds.). (2025). Educator reflections: The power of our stories. MT Open Press, Middle Tennessee State University. https://doi.org/10.56638/mtopb00325

Chapter 2-4

The Challenge and Joy of Growing as a Mathematics Teacher

Terrance Quinn

> Sir Thomas More: Why not be a teacher? You'd be a fine teacher; perhaps a great one.
> Richard Rich: If I was, who would know it?
> Sir Thomas More: You; your pupils; your friends; God. Not a bad public, that.
>
> Robert Bolt, A Man for All Seasons, A Play in Two Acts

This chapter entry is mainly intended for those considering becoming a mathematics teacher in elementary school through K–12 or beyond. I share and reflect on a few aspects of my path to teaching mathematics. I found that to grow in competence as a teacher, I needed to continue my growth and development in mathematics. In a manner that will be indicated later, I also found that I improved as a teacher by making progress in identifying elements of my growth and development in mathematics. I have found that even modest progress along these lines has been remarkably fruitful.

I slowly grew into loving mathematics. A serious interest and a habit of study were already there when I chose it as my major. For me, doing mathematics has never not required a sustained effort. A solution is "clear" only when it becomes so. And there is no formula for when that might happen. But I have found that there can be, as it were, types of joy that have sustained me prior to, during, and consequent to struggling with a problem and eventually "getting it." Over time, I began to appreciate the beauty of mathematical understanding.

Throughout my student years, I also grew to enjoy teaching. Sometimes, this was in a formal setting, such as being a teaching assistant or, later, when I was a graduate student, being the instructor for a course. But often enough, the setting was informal and included, for example, helping students who were at earlier stages in their mathematics and science programs. I was becoming aware of the pleasure of sharing satisfying moments of insight and development in technique. At the same time, however, my main focus remained centered on the possibility of eventually doing research in mathematics and, perhaps, in the philosophy of science.

Initially, unbeknownst to me, my first university tenure-track position started changing that by nudging me toward the possibility of a larger horizon with kinds of growth that I had not yet envisaged. This growth began mundanely, with me having a higher teaching load than I would have chosen for myself at that stage of my career. Teaching was enjoyable, but it was not my main goal, and it took time away from being able to do research. But I was dedicated to my job. I thus gladly gave myself to the task. And so, it started. Not part of my earlier plan, but slowly, with many missteps along the way and not without considerable labor, I began to grow as a teacher.

My commitment to improving as a teacher started to deepen, I might say, exponentially. I began to inquire into what worked and what did not. I soon realized that these lines of inquiry give rise to fundamental and challenging questions. What does "it worked" or "it did not work" mean? What is "it"? Is "it" a matter of helping students succeed on tests and exams? Of course, there is more to it than that. What was on offer from theories of mathematical learning provided little help. They were mainly developed on the hypotheses of speculative models in general terms remote to human experience. (Unfortunately, I have found that, so far, that remains the dominant ethos in the scholarship of mathematics education.) I needed to learn how to better teach specific results, working with these students in this course, in this program, in this and that instance, in this formula, and in that theorem. Since part of my teaching assignments often included teaching students in the mathematics education programs, I also needed to improve my grasp of teaching others how to teach mathematics.

While faced with these challenges, I realized that a crucial source of reflection on learning and pedagogy in mathematics is my own experience in the subject. Following up on this, the need for and the possibility of what is now called tandem method was becoming evident. I am referring to the possibility of attending, as needed, to two distinct but never separate sources in experience. I took help from the 1975 edition of McShane (2021), which, instead of saying "tandem method," speaks of "dual interest" (McShane, 2021, p. 18). For this chapter, then, the expression "tandem method" can be taken descriptively, that is, in the sense that one can have two focuses of attention. Its origins in the philosophy of science are in the literature that is centered on the works of Bernard Lonergan (1904–1984). (See, for instance, [Duffy, 1996, p. 240]). I started to learn how to advert to, distinguish, and relate not only, as is normal, results of my mathematical thinking (e.g., particular concepts and terms, formulas, theorems), but also distinct sources of these results. For instance, I began to attend to and distinguish shifts in my cumulative contextualizing, diagramming, symbolizing, and technique, and in my wondering, "What is it?" and "What to do?," not to mention my desire for continued growth in understanding.

As you might suspect, and as I found, tandem method is concrete. Part of the challenge is to neither begin with nor attempt to impose speculative models, conceptual orderings, or systems, let alone philosophical views developed from dubious analogies of human understanding. Instead, I began adverting to details of what I do when I do mathematics. The scope of tandem method, then, does not exclude concepts, axioms, and postulates. It is based, rather, on what I, and each of us, find by attending to instances in, and details of, our own inquiry and insight in mathematics, which include, among other things, the emergence of concepts and the development of axioms.

To be sure, this is all too brief. A short chapter entry is not the place to provide a decent introduction to tandem method in mathematics. (For introductory presentations, see Benton & Quinn, 2022; Quinn et al., 2020; Quinn, 2024). However, for the prospective mathematics teacher reading this, glimpses of the potential fruitfulness of the method can be had by considering a familiar result in K–12 curricula.

I am referring to what is often called the Pythagorean theorem. Although, independently of Pythagoras (c. 570 B.C.E.– 495 B.C.E.), special cases were discovered long beforehand in Mesopotamia. (See, for instance, Katz, 2009, pp. 17–22). Early versions were also known in ancient China and probably elsewhere. In the *Elements*, Euclid (c. 300 B.C.E) developed a specialized version of the theorem in an axiomatic system.

In any event, at this stage, the formula $c^2 = a^2 + b^2$, is familiar, as is what it refers to, namely, a right-angled triangle with hypothenuse c and sides a and b. But what does the formula mean? More to the point, what do I mean, but eventually also what might you mean when referring to a diagram, uttering (written phonetically)

[si skwɛrd ˈikwəlz ə skwɛrd plʌs bi skwɛrd]

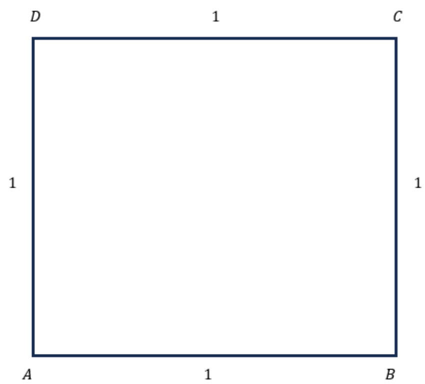
and writing the symbols '$c^2 = a^2 + b^2$'? We can imagine and name a right-angled triangle. We can know how to use the formula $c^2 = a^2 + b^2$ by plugging in numbers and perhaps also algebraically in conjunction with other formulas. But where does the formula come from in the first place?

Most current textbooks (from elementary school onward) begin with "the answer" and thus provide little or no direct assistance in that regard. Wonder and insight are effectively short-circuited. Perhaps at some point in high school, you worked through Euclid's proof of Proposition 47 of Book I of the *Elements* (Heath, 2012). But you might notice that this does not get us off the hook. The proposition also starts with the answer. It states that "[i]n right-angled triangles, the square on the side tending the right-angle is equal to the [sum of the] squares on the sides containing the right-angle" (Heath, 2012, p. 349). The statement, then, explicitly presupposes a prior discovery. But if you or I discover a possible mathematical relationship, we can also ask, "Is it so? Is it not so?" Notice that we cannot reasonably answer this type of question with a formula, definition, or result that proceeds from insight in a "What is it?" mode of inquiry. But that is not a problem. For as experience shows, the proof of Proposition 47 is not for that purpose. Rather, it facilitates the occurrence of a different type of understanding wherein one grasps a sufficiency of evidence so that we can assent provisionally to the proposition in the context of the Euclidean axiomatic system.

Let us try another approach to help bring out some of the key and core issues: instead of beginning with an answer, let us start with a question. Consider a square whose sides are of length 1 (cm, inch, or whatever). The question is, how do we "double the square"? In other words, what are the dimensions $c \times c$ of a square so that the area is twice that of the square whose dimensions are 1×1? Geometrically, how do we determine c so that $c^2 = 2(1) = 2$?

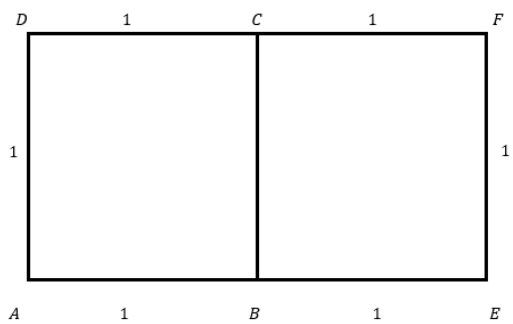
In fact, this is a famous problem that goes back to ancient times. (See, for instance, the dialogue in Plato's book called *Meno*.) But we can pose the problem and solve it here, together. Consider a square, then, such as represented in Figure 1.

Figure 1: A unit square $ABCD$, drawn by the author



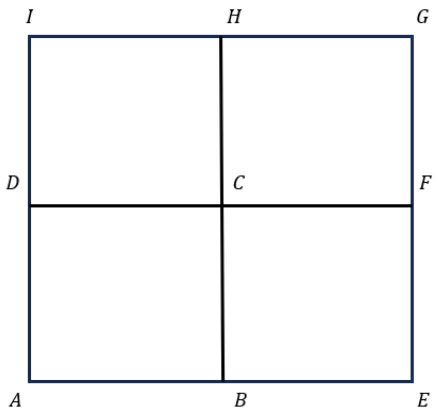
Our goal is to double the area, so one way to begin might be to imagine or draw two unit squares, thus producing rectangle $AEFD$ (Figure 2).

Figure 2: Mere duplication of a unit square, drawn by the author



This does not yet get us to a solution to the problem, but it starts us off in what perhaps is a promising direction for increasing area. Might we add to the diagram? For instance, if we duplicate the rectangle $AEFD$, we get a larger square $AEGI$ (Figure 3).

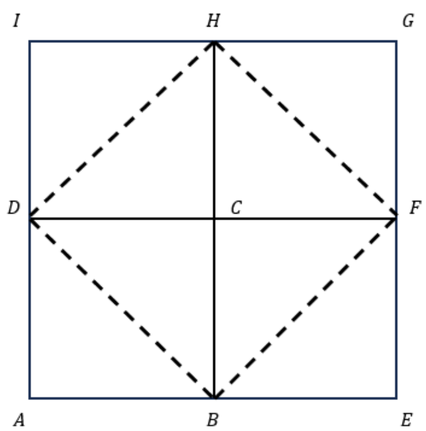
Figure 3: Four unit squares, drawn by the author



The area of the square $AEGI$ is four times the area of the unit square $ABCD$. This is more than we need. But since it is four times the area of the unit square, can we, perhaps, find a square within the larger square that solves the problem?

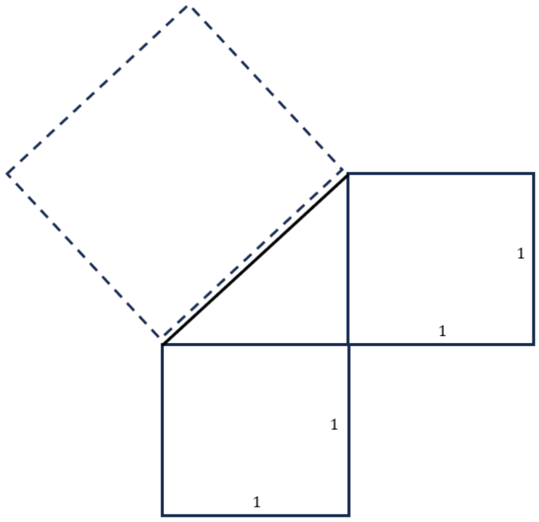
At this stage, the larger square is partitioned by verticals and horizontals. But we could, for instance, introduce a diagonal DB. This produces a right angle triangle DAB, where the perpendicular sides DA and AB are each of unit length. With that clue, you may have now solved the problem. But let us continue with the idea and so introduce more diagonals, namely, BF, FH, and HD, as in Figure 4.

Figure 4: Doubling the unit square, drawn by the author



Behold! A square that is based on the hypothenuse of a unit right angle triangle has twice the area of a unit square. This solution to the problem is represented in Figure 5.

Figure 5: A square based on the hypothenuse of a unit right angle triangle doubles the unit square, drawn by the author



In this way, we have had a key insight and made a beginning in understanding the Pythagorean theorem for ourselves. Notice that we also have obtained the familiar geometric representation indicated in the statement of Euclid's Proposition 47, not as a prescribed diagram but rather *as a consequence* of our insight. With numbers representing lengths and products representing areas, we get $\left(\sqrt{2} \times \sqrt{2}\right) = (1 \times 1) + (1 \times 1)$. Or, more compactly, we can write $\left(\sqrt{2}\right)^2 = 1^2 + 1^2$.

If you have worked through this example, you might pause with me now to notice that something remarkable has happened. We have gotten a hold of something that is in addition to merely imagining and naming and being able to "plug and chug" numbers into a formula given by a textbook. Our understanding is of a possible correlation between three imagined lengths. (More precisely, we have grasped a possible correlation of correlations of correlations. But that is an advanced result [Quinn, 2024]). You might also notice that that correlating in us emerged while attending to and wondering about, but is not reducible to, diagrams and symbols.

Now that we have worked through an example, let us think about how our experience can help in teaching. If you have had the key insight for doubling the square (and perhaps gone on to a similar key insight for the more general case expressible by the formula $c^2 = a^2 + b^2$), you will be able to comfortably talk at length to help others with questions from many perspectives. If, however, when teaching, you find that you are only able to repeat a formula or diagram, or merely reproduce the steps of an established proof, or substitute numbers into a formula, then that is evidence that you are missing a key insight. But do not worry. That happens to every student and every teacher at every level. We are all works in progress. When the need arises, we can go back and, as it were, fill in the gaps or, rather, meet the need for understanding. Thusly enlightened, and then, using tandem method, we will be able to identify details of how we got that further insight. Our ability to help others understand will then also be boosted considerably.

I have provided the thinnest of glimpses of what I have found to be a richness, control, and fruitfulness that can emerge through implementing tandem method in learning and teaching elementary mathematics. Now, imagine continuing as much as possible with the double focus. There are questions and key insights all along the way. And so it has been for me. In good measure, spurred on by a deepening desire to be a good teacher, I started to make progress in a concrete approach to learning about my learning in mathematics. A consequence of this has been a stable basis for ongoing growth in being able to identify elements and cumulative "layerings" in my mathematical development, evidenced in detail in instances in my experience. I should also mention that progress in tandem method has been providing me with a foundation from which to evaluate classroom methodologies that regularly change.

As it turned out, then, without having to give up scholarship, teaching became a second and mutually enriching vocation. The task of teaching pushed me to the need for a double focus or tandem method (that is, the need to make ongoing progress in mathematical understanding and in understanding my ongoing progress in mathematical understanding). While always provisional, my growth in tandem method has also provided me with a verifiable basis for helping others grow in understanding mathematics. I have found that all of this has been both challenging and an ongoing source of joy.

REFERENCES

Benton, J., & Quinn, T. (2022). *Journeyism: A handbook for future academics* (1st ed.). Island House Press.

Bolt, R. (1960). A man for all seasons. Canadian Educational Edition, Notes and Questions by Procunier, E. R. Bellhaven House Limited, Scarborough, Ontario, Canada.

Heath, T. H. (2012). *The thirteen books of the elements: Vol. 1: Books 1-2* (2nd ed.). Dover Publications Inc.

Duffy, J. D. (1996). The ethics of Lonergan's existential intellectualism [Fordham University]. https://research.library.fordham.edu/dissertations/AAI9628334

Katz, V. J. (2009). *A history of mathematics, an introduction* (3rd ed.). Addison-Wesley.

McShane, P. (2021). *Wealth of self and wealth of nations: Self-axis of the great ascent* (2nd ed.). Axial Publishing.

Plato. (1980). *Meno*. (G.M.A. Grube, Trans.) Hackett Pub. Co.

Quinn, T. (2024). *(Growing)^2 in Mathematics*. Island House Press.

Quinn, T., Boudhraa, Z., & Rai, S. (2020). Supplement on mathematical understanding. In *Basic Insights in Vector Calculus*. World Scientific.